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Statistical properties of classical gravitating particles in (2+1) dimensions

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Abstract. We report the statistical properties of classical particles in (2+1) gravity as resulting from numerical simulations. Only particle momenta have been taken into account. In the range of total momentum where thermal equilibrium is reached, the distribution function and the corresponding Boltzmann entropy are computed. In the presence of large gravity effects, different extensions of the temperature turn out to be inequivalent, the distribution function has a power law high-energy tail and the entropy as a function of the internal energy presents a flex. When the energy approaches the open universe limit, the entropy and the mean value of the particle kinetic energy seem to diverge.

1. Introduction. The inclusion of gravitational effects in thermodynamics has important consequences at the cosmological level. There is a particular interest in this subject also because gravity deeply modifies certain basic features of non-gravitating systems; for instance, in the presence of gravitational interactions one cannot define the thermodynamic limit in the standard way, one may find bounds for the value of the entropy, the additivity of the thermodynamic potentials is no longer valid in general, etc. In this article we explore the statistical properties of a classical gravitating gas in (2+1) dimensions with vanishing cosmological constant. The gravitational dynamics in (2+1) dimensions is rather simple but it is not trivial, so this is one of the simplest models in which one can look for new phenomena.

Our results are based on numerical simulations. We have considered a set of classical pointlike particles, with random initial momenta, moving on a spatial surface of topological type either $\mathbb{R}^2$ or the sphere S^2 for a finite time interval. We followed only the evolution of the particle momenta. Let G be the gravitational constant and U the total energy of the system. When $4GU < 1$, the system reaches thermal equilibrium. Whereas, for space-like total momentum or when $4GU > 1$, the numerical analysis shows the presence of dynamical instabilities; jets of particles appear with no-limits increasing energy.

In the case $4GU < 1$, the resulting distribution function in momentum space f is numerically produced and the corresponding Boltzmann entropy, $S \propto \int f \log f$, is computed as a function of the total energy U of the system. If $4GU \ll 1$ the gravitational effects can be neglected and the entropy S shows the classical $\log U$ behaviour. On the other hand, when the value of the energy approaches the limit $4GU = 1$, gravity has a strong influence on the dynamics of the particles and the value of S seems to diverge. The implications of this behaviour on the heat capacity of the gas are considered. Finally, we compare various definitions of temperature which turn out to be inequivalent in the energy range of large gravity effects.

2. Classical gravitating particles. The peculiar properties of (2+1) gravity have been described in Ref. [1,-,4]. In any (2+1) spacetime M containing classical pointlike particles, all the curvature is located on the worldlines of the particles. Each particle moves in a locally flat surrounding spacetime. Since the isometry group of flat spacetime is isomorphic with the Poincaré group $ISO(2,1)$, both the kinematics and the laws of the dynamics are covariant with respect to transformations of the Lorentz group $SO(2,1)$.

A rather simple description of the spacetimes containing N pointlike gravitating particles has been produced by 't Hooft [5,6]. This description is given by the linear evolution of Cauchy surfaces which are tiled by spatial planar polygons; the time evolution also includes modifications of the tiling combinatorics according to codified transition rules [6]. A Lorentz frame is associated to each polygon with local coordinates (t, x, y) and metric $ds^2 = dt^2 - dx^2 - dy^2$; with respect to this frame, the interior of each polygon belongs to the spatial $t =$ constant surface. The extrinsic

curvature is vanishing in the interior of each tile and is singular on the edges. Each particle is placed in one corner of a polygon; this corner is the intersection point of two consecutive edges which must be identified.

At a given initial time $t = t_0$, one can always introduce a tessellation in which all the particles are placed at the corners of one polygon $\mathcal{P}(t_0)$; let us denote by LS the Lorentz system which is associated with $\mathcal{P}(t_0)$. The initial values $\{ p^a_{(i)} \}$ (with $i = 1, 2, ..., N$) of the particle momenta are defined with respect to the frame LS. In addition to $\mathcal{P}(t_0)$, the $(t = t_0)$ tessellation may possibly contain several polygons; the complete description of the initial configuration includes all the lengths and velocities of the polygon edges.

During a short time interval after $t = t_0$, one has a "linear" evolution. Because of the constant velocities of the edges, the length of each edge is a linear function of time, the angles between edges and the particle momenta are constant in time. When the length of one edge vanishes or when one of the corners hits an edge, one has a transition in which the structure of the tessellation undergoes a local change. So, the nontrivial part of the evolution consists of a sequence of transitions; in each transition the shape and the number of polygons, together with the velocities of the edges, are in general modified according to a given set of rules [6].

After a finite time interval $\Delta t = t_1 - t_0$ in which no big-crunch occurs, one finds a tessellation in which the particles are in general placed at the corners of different polygons. In order to compare the initial $(t = t_0)$ and the final $(t = t_1)$ values of the particle momenta, we need to refer these values to a unique coordinate system; we shall choose the Lorentz frame LS as reference system.

Let Q_0 be a point in the interior of $\mathcal{P}(t_0)$; we shall denote by Q_1 the point in M which has the same spatial coordinates as Q_0 and has time coordinate t_1 with respect to the frame LS. We assume that Q_1 does not belong to the worldline of a particle; for, if this is the case, we can simply modify the spatial position of Q_0 so that the corresponding Q_1 does not belong to a worldline. In a neighbourhood $\Omega \subset M$ of Q_1 the spacetime is flat and one can introduce local coordinates which are adapted to the Lorentz system LS. The piece in Ω of the spatial surface $t = t_1$ can be extended in M and one will eventually reach the worldlines of the particles. This extension is not unique, in general. In order to remove all ambiguities, we shall require that the extended surface $t = t_1$ is star-shaped with respect to the geodesic lines of minimal (spatial) distance emerging from Q_1; this simply means that if a point Q belongs to the surface, then all the points of the (minimal-length) geodesic connecting Q with Q_1 also belong to the surface. This extension of the spatial surface $t = t_1$ determines a polygon $\mathcal{P}(t_1)$ with the property [6,7] that each particle is placed at a corner of $\mathcal{P}(t_1)$. Moreover, one can introduce a new $(t = t_1)$ tessellation of space containing the tile $\mathcal{P}(t_1)$. With respect to the Lorentz frame LS of $\mathcal{P}(t_1)$, which is the same frame of $\mathcal{P}(t_0)$, the final $(t = t_1)$ values of the particle momenta will be denoted by $\{ k^a_{(i)} \}$.

To sum up, with respect to a fixed Lorentz system, the time evolution results in a

modification of the particle momenta

$$U(t_1,t_0)\ :\ \{\,p^a_{(i)}\,\}\ \longrightarrow\ \{\,k^a_{(i)}\,\}\qquad ,$$

where the map $U(t_1,t_0)$ depends on the initial $(t=t_0)$ data. We shall conclude this section by recalling the general structure of $U(t_1,t_0)$.

Consider a set of oriented closed paths $\{\,\gamma_i\,\}$ on $\mathcal{P}(t_0)$ which are based on the point Q_0 and have no intersection with the particle worldlines. Let the loop γ_i be simply linked with the worldline of the i-th particle; the paths $\{\,\gamma_i\,\}$ are closed because some of the edges of $\mathcal{P}(t_0)$ must be identified in the tessellation. The effects of a parallel transport of a vector along the loop γ_i can be described by means of a Lorentz transformation $H(\gamma_i)$ acting on the tangent space in Q_0. Since M is flat in a neighbourhood of Q_0, the Lorentz frame LS canonically defines a reference system in the tangent space at Q_0. The spacetime curvature is concentrated on the particle worldlines, so $H(\gamma_i)$ is invariant under smooth deformations of γ_i. Moreover, one has $H(\gamma_i^{-1}) = H^{-1}(\gamma_i)$ where γ_i^{-1} represents γ_i with reversed orientation. Now, with a suitable choice of the loops $\{\,\gamma_i\,\}$ and a given (clockwise) prescription for the loop orientations, one finds [6]

$$H(\gamma_i)\ =\ \exp\left(\,8\pi G\,p^a_{(i)}J_a\,\right)\ =\ \exp\left(\,\widehat{p}_{(i)}\,\right)\quad ,$$

where $\{\,J_a\,\}$ are the generators of the Lorentz group and $\{\,p^a_{(i)}\,\}$ are the components of the momentum of the i-th particle with respect to the reference system LS. Since all the particles are placed at the corners of the polygon $\mathcal{P}(t_0)$ whose boundary -without edges identifications- has the topology of S^1, an ordering of the particles turns out to be fixed modulo cyclic permutations; let us assume that the natural ordering of the index $i=1,2,...,N$ corresponds to consecutive particles on the boundary of $\mathcal{P}(t_0)$. The initial $(t=t_0)$ particle data can conveniently be expressed by means of an ordered set I of $SO(2,1)$ group elements

$$I\ =\ \{\,H(\gamma_1)\,,\,H(\gamma_2)\,,\ldots,\,H(\gamma_N)\,\}\ =\ \{\,e^{\widehat{p}_{(1)}}\,,\,e^{\widehat{p}_{(2)}}\,,\ldots,\,e^{\widehat{p}_{(N)}}\,\}\qquad .$$

We now need to introduce the Lancaster-Sasakura representation [8] of the braid group B_N acting on I; the generator σ_j of B_N (with $j=1,2,...,N-1$) is represented by

$$\sigma_j\ :\ \{\,e^{\widehat{p}_{(1)}},\ldots,\,e^{\widehat{p}_{(N)}}\,\}\ \mapsto\ \{\,e^{\widehat{p}_{(1)}},\ldots,\,e^{\widehat{p}_{(j-1)}}\,,\,e^{\widehat{p}_{(j+1)}}\,,\,e^{-\widehat{p}_{(j+1)}}e^{\widehat{p}_{(j)}}e^{\widehat{p}_{(j+1)}}\,,\cdots,\,e^{\widehat{p}_{(N)}}\,\}\,.$$

Let us denote by F the ordered set of $SO(2,1)$ group elements

$$F\ =\ \{\,e^{\widehat{k}_{(1)}}\,,\,e^{\widehat{k}_{(2)}}\,,\ldots,\,e^{\widehat{k}_{(N)}}\,\}\qquad ,$$

where $\{\,k^a_{(i)}\,\}$ are the final $(t=t_1)$ particle momenta. One can show [6,-,9] that the time-evolution map is given by an element $g(t_1,t_0)\in B_N$ in the Lancaster-Sasakura representation

$$U(t_1,t_0)\ :\ I\ \rightarrow\ F\ =\ g(t_1,t_0)\,I\qquad . \tag{2.1}$$

The particular element $g(t_1, t_0) \in B_N$ which enters equation (2.1) depends on the initial positions and velocities of the particles. Any element of the braid group can be written as an ordered product of generators. So the time evolution can be interpreted as the result of a certain number of consecutive elementary processes; each elementary process is associated with a generator of the braid group.

3. Computation rules. For any specific choice of the initial lengths and edge velocities of the starting tessellation, 't Hooft rules determine a unique time evolution. With a large number of particles, the complete evaluation of the exact microscopic state of the system is rather laborious [6]. On the other hand, we are interested in the thermodynamic properties of a large number N of gravitating particles. Therefore we shall introduce certain approximations which simplify the computation of the macroscopic variables. The main idea consists of adopting a iterative method -based on some random process- to determine the Braid group element $g(t_1, t_0)$ which specifies the evolution (2.1) of the particle momenta.

If one has a 3-dimensional classical gas in standard conditions of temperature and pressure, one can approximate the molecular motion by a sequence of diffusion processes among randomly chosen pairs of molecules. This is not an accurate approximation at the microscopic level but turns out to be a rather good method to evaluate macroscopic variables. Similarly, for the 2-dimensional gravitational gas we will approximate the time evolution by a sequence of "scattering processes" involving randomly chosen pairs of particles.

Suppose that a and b are two integers which have been randomly chosen among the set $\{ 1, 2, ..., N \}$ with the condition $a \neq b$. Suppose that, for instance, $a < b$. One can then define a particular element $h(a, b)$ of the braid group B_N which is the product of $(b - a)$ generators

$$h(a, b) \; = \; \sigma_{b-1} \cdots \sigma_{a+1} \cdot \sigma_a \qquad .$$

The elementary process associated with the integers a and b can be interpreted as a scattering process involving the a-th and b-th particles. After this scattering, the particle momenta are described by the new ordered set of $SO(2, 1)$ elements

$$I' \; = \; h(a, b)\, I \qquad , \tag{3.1}$$

or

$$I' \; = \; h^{-1}(a, b)\, I \qquad . \tag{3.2}$$

Possibilities (3.1) and (3.2) are assumed to have equal probabilities to occur. After this first step, one has simply to repeat the procedure: one determines two new random numbers a' and b' and applies $h(a', b')$ (or $h^{-1}(a', b')$, with probability 1/2) to the ordered set I' in order to find I'', and so on.

One can use a simpler recipe to approximate the time evolution. At each step of this new recipe, one randomly determines an integer $a \in \{1, 2, ..., N - 1\}$ and the

modification of the particle momenta is described, with equal probabilities, by the generator σ_a or its inverse σ_a^{-1} in the Lancaster-Sasakura representation. We have verified that, in all the considered examples, both methods lead to the same conclusions.

Our computation rules only concern the values of the particle momenta; the spatial positions of the particles and their mutual distances have been ignored. In the ordinary 3-dimensional case of a gas inside a finite volume, one can safely ignore the spatial positions of the molecules because the distribution of equilibrium is in fact homogeneous. So, as long as the gas is approaching the thermal equilibrium, the actual position of each molecule plays no role in the evaluation of the macroscopic variables. For this reason, the assumption of a homogeneous equilibrium distribution could be introduced to justify our computation rules. Actually, the 2-dimensional gravitating gas that we consider is not evolving inside a container with fixed walls; consequently, the equilibrium particle distribution -when it exists- is not necessarily homogeneous in space. Yet, the computation rules that we use to approximate the evolution of the particle momenta are expected to be valid anyway because each elementary process involving two gravitating particles in (2+1) dimensions does not depend on the distance between these particles.

4. Conservation law. Since we ignore the spatial positions of the particles as well as the whole "extension" of the system, we can only compute how the thermodynamic potentials depend, for instance, on the internal energy U of the system. So we need to give a definition of U and discuss the associated conservation law.

The components $p^a = (p^0, \vec{p})$ of the momentum of a classical pointlike particle of mass m can be determined by measuring the velocity of the particle with respect to a local Lorentz frame. Because of Lorentz covariance, the energy p^0 and the spatial components $\vec{p}$ of the momentum have the usual dependence on the mass and on the velocity of the particle. One has $(p)^2 = p^a p_a = m^2$. For a system of $N > 1$ gravitating particles, the sum of the energies of the particles is not a conserved quantity. In the presence of gravitational interactions, the ordinary conservation law for the 3-momentum gets modified; the new conservation law [7] for the 3-momentum is Lorentz covariant and can be expressed in terms of $SO(2,1)$ group elements. Since the time evolution of the particle momenta is given in equation (2.1), it is easy to verify that, whatever $g(t_1, t_0) \in B_N$ might be, the composite element

$$e^{\widehat{p}_{(1)}} \cdot e^{\widehat{p}_{(2)}} \cdots e^{\widehat{p}_{(N)}} \;=\; e^{\widehat{p}_{(tot)}} \;=\; \exp\left(\, 8\pi G \, p^a_{(tot)} J_a \,\right) \tag{4.1}$$

is conserved; $\{\, p^a_{(tot)} \,\}$ are the components of the total 3-momentum of the N particles. In (2+1) dimensions the gravitational field does not describe propagating degrees of freedom; indeed, the total 3-momentum only depends on the variables associated with the particles. No Newtonian or potential energy is present in (2+1) dimensions, and in fact $\{\, p^a_{(tot)} \,\}$ do not depend on the positions of the particles. As expected, in the $G \to 0$ limit, expression (4.1) reproduces the usual additive conservation law of 3-momentum which is valid in ordinary Minkowski space.

The total 3-momentum admits an intrinsic definition. Consider an oriented loop γ in M based on the fixed point $Q_0 \in M$ and let Σ be an oriented disc in M whose boundary coincides with γ. Each particle worldline has a natural orientation and its intersection points with Σ have definite signatures. We will say that a given worldline is enclosed by γ when the algebraic sum of its intersections with Σ is equal to unity. Suppose now that all the worldlines of the particles are enclosed by γ; the path-ordered integral of the spin connection (which must be defined in a smooth coordinate system) along γ determines the total 3-momentum. In facts, the $SO(2,1)$ element which describes the effects of a parallel transport of vectors along γ is given by

$$H(\gamma) \;=\; \exp\left(\, 8\pi G\, p^a_{(tot)} J_a \,\right) \qquad . \tag{4.2}$$

When the total 3-momentum is timelike, one can find a Lorentz transformation Λ such that

$$\Lambda \cdot \exp\left(\, 8\pi G\, p^a_{(tot)} J_a \,\right) \cdot \Lambda^{-1} \;=\; \exp\left(\, 8\pi G\, U J_0 \,\right) \qquad , \tag{4.3}$$

where U is called the internal energy of the particles system. The generator of rotations J_0 has integer eigenvalues and then $8\pi GU$ is apparently defined only modulo 2π. Actually this is not the case; the value U of the internal energy is well defined in the entire real axis. Indeed, consider a given parametrization of γ with variable $\tau \in [0,1]$; since the spin connection takes values in the algebra of the Lorentz group, the path-ordered integral of the spin connection in the interval $[0,\tau]$ defines a continuous map $\psi : \tau \mapsto \psi_\tau \in \widetilde{SO}(2,1)$ where $\widetilde{SO}(2,1)$ is the infinite cyclic covering of $SO(2,1)$ [10]. By construction, one has $\psi_0 =$ identity and the projection of ψ_1 into $SO(2,1)$ coincides with $H(\gamma)$. The fact that the value of U is well defined in the whole real axis should not be surprising; something similar occurs with the time defined by a watch. Even if the position of the minute-hand is specified by an angle which is defined modulo 2π, there is no ambiguity at all in determining, by continuously observing the watch, the time interval elapsed with respect to a given moment.

5. Numerical results. In order to complete the description of the computation rules, we need to discuss the specification of the initial data which consists of three steps:

1. it is assumed that all the particles have the same mass m; a set of values $\{\, q^a_{(i)} \,\}$ for the particle momenta is fixed by choosing the spatial components $\{\, \vec{q}_{(i)} \,\}$ randomly within a finite domain;

2. for a given set of values $\{\, q^a_{(i)} \,\}$ determined in point 1, the corresponding total 3-momentum is computed by means of the relation (4.1);

3. if the total momentum is time-like, by means of a Lorentz transformation Λ the particles system is taken to its rest frame. Λ is fixed according to equation (4.3) which also determines the internal energy U of the system. The set of particle

momenta with respect to the system rest frame $\{ p^a_{(i)} = \Lambda^a{}_b q^b_{(i)} \}$ represents the ordered set of initial particle momenta on which the time evolution acts.

In the various numerical simulations that we have performed, the number of particles has been chosen within $10^2 \leq N \leq 10^3$ and the values of the masses belong to the range $0 \leq m \leq 10^{-2}/4G$. Since the total 3-momentum has been determined by means of equation (4.1), the values of the mass and of the initial momentum of each particle have been chosen to be "small enough" so that no $1/4G$ ambiguity resulted in the computation of the total energy. Instead of performing the Lorentz transformation mentioned in point 3, in several occasions we have used a different method to bring the system in its rest frame; namely, to the set of the initial N particles we have simply added a new particle with a suitably chosen momentum so that the resulting spatial components $\vec{p}_{(tot)}$ vanish. This last method is convenient because it avoids strong correlations among initial particle momenta.

5.1. Open universe. We have done simulations consisting of approximatively 4×10^9 random interactions and we have verified that, as long as $U < 1/4G$, the simulations always reached a stationary state within the first 10% of the interactions, which we excluded from the final data. The numerical results are expressed by means of the equilibrium distribution function $f(\vec{p})$ such that

$$dN \;=\; f(\vec{p})\; d^2p$$

represents the particle density in momentum space. The normalization condition is

$$\int d^2p\; f(\vec{p}) \;=\; N \quad .$$

Since $f(\vec{p})$ is isotropic, it only depends on $|\vec{p}\,|$; so it is convenient to introduce a distribution function $\rho(p^0)$ for the energy $p^0 = \sqrt{m^2 + |\vec{p}\,|^2}$ of a single particle:

$$\rho(p^0) \;=\; \frac{2\pi\, p^0}{N}\, f(|\vec{p}\,|) \;=\; \frac{2\pi\, p^0}{N}\, f(\sqrt{(p^0)^2 - m^2}) \quad .$$

The ρ normalization is

$$\int_m^{+\infty} dp^0\; \rho(p^0) \;=\; 1 \quad .$$

When $U < 1/4G$, we observed that the system always reaches a stationary state. Two examples of computed distribution function ρ are shown in Figure 1 and 2. In both cases, $N = 10^3$ and $m = 0$. Figure 1 refers to the case in which $4GU = 1.296 \times 10^{-1}$, whereas in Figure 2 one has $4GU = 9.9968 \times 10^{-1}$. In all the figures, energies and momenta are expressed in units of $1/8\pi G$. The qualitative behaviour of the gravitating gas does not depend on the value of m provided $Nm < 1/4G$.

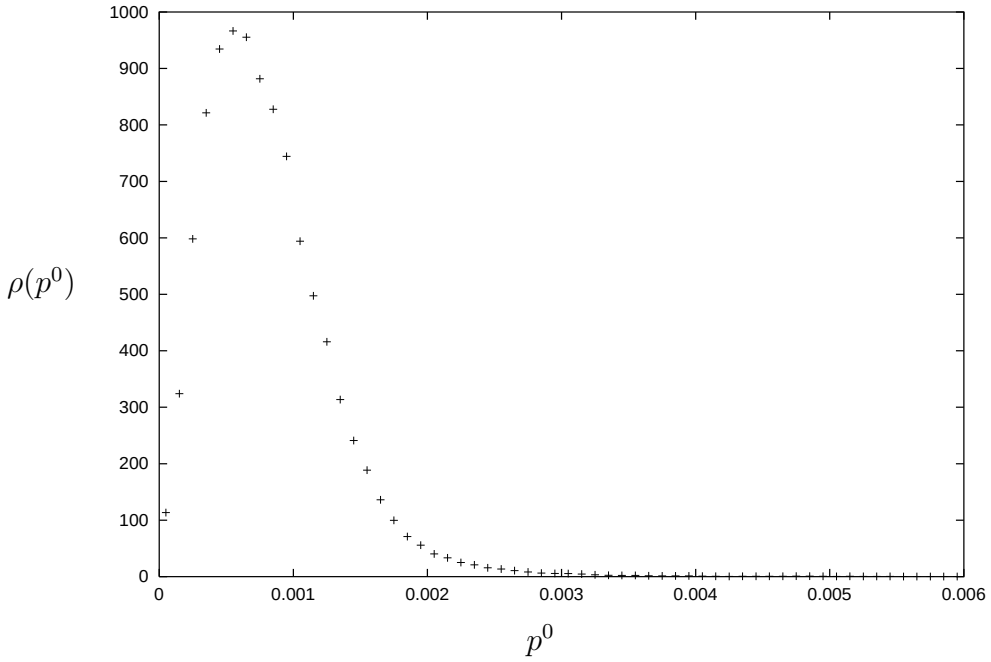

Figure 1: plot of $\rho(p^0)$ at $N = 1000,\ m = 0,\ 4GU = 1.296 \times 10^{-1}$

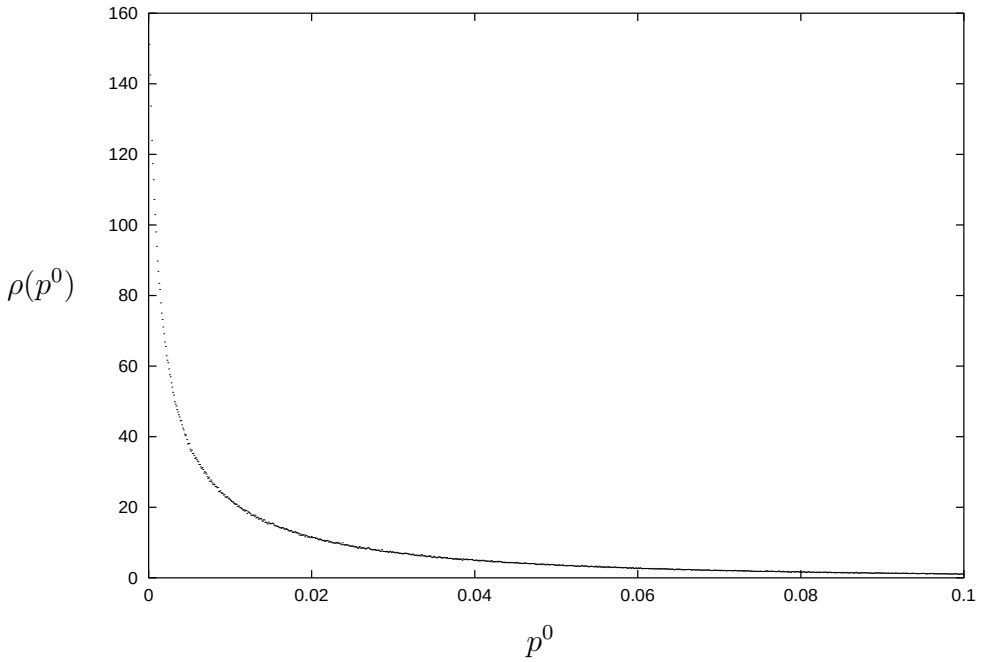

Figure 2: plot of $\rho(p^0)$ at $N = 1000,\ m = 0,\ 4GU = 9.9968 \times 10^{-1}$

In the low energy limit $U \ll 1/4G$ the equilibrium distribution $f(\vec{p})$ is well fitted by the Boltzmann distribution that we shall derive later. This is in agreement with the fact that, for $G \to 0$, the energy conservation law reduces to the classical one.

As U grows, $f(\vec{p})$ differs from the Boltzmann distribution substantially. The high-energy tail of $\rho(p^0)$ is rather well fitted by the function $a/(p^0 + b)$ with a truncation at a certain $p^0 = \overline{p}^0(U)$.

A new phenomenon happens when the value of U approaches $1/4G$: the whole gas strongly correlates, all the particle momenta tend to become parallel and the overall direction fluctuates randomly as the system evolves. Figure 3 shows the mean value $\langle\, \vec{p}\,\rangle$ of $\vec{p}$ in a finite time interval for $4GU$ almost equal to unity. The time evolution presents large fluctuations in $|\vec{p}\,|$ which appear as spikes in the picture.

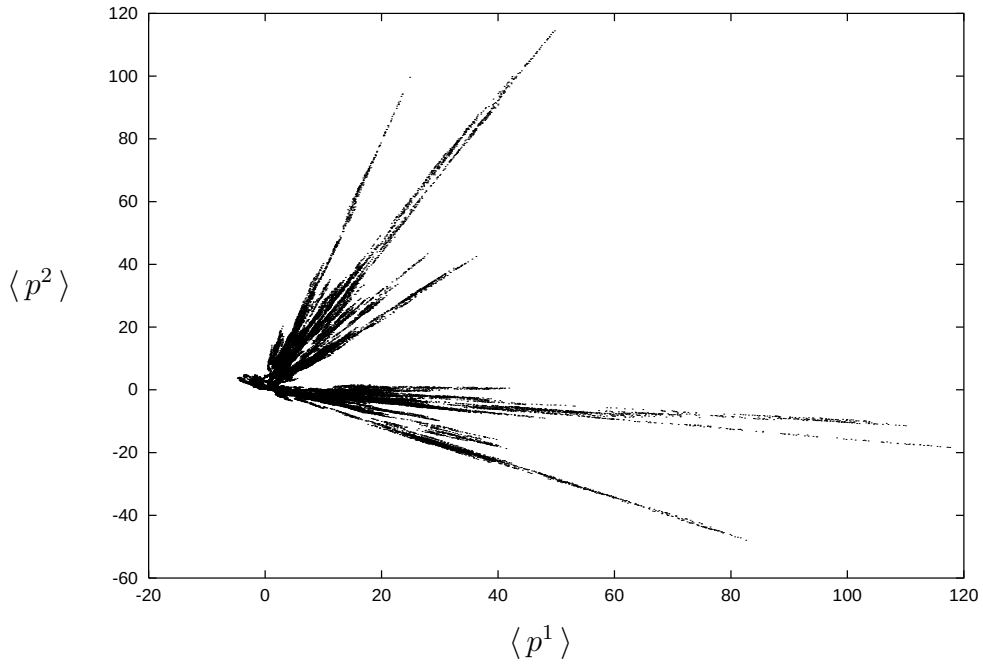

Figure 3: plot of $\langle\, \vec{p}\,\rangle$ at $N = 100,\ m = 7.161 \times 10^{-3},\ 4GU = 9.9980 \times 10^{-1}$

5.2. Spacelike total momentum. In this case, numerical simulations show that the system is unstable and no equilibrium distribution exists. During the evolution, the energies of the particles tend to increase overcoming whatever fixed value. The way energies diverge is quite peculiar: at first, all energies tend to increase slowly and, after a while, they grow faster and faster. The higher the energy is, the quicker it increases (positive feedback) so nothing can prevent energies to diverge. At this stage, Gott pairs are created and a "jet" appears. Two consecutive particles form a Gott pair [11] when the non-Abelian composition p of their momenta,

$$\exp\left(\widehat{p}_{(1)}\right)\cdot\exp\left(\widehat{p}_{(2)}\right) \;=\; \exp\left(\widehat{p}\,\right) \quad ,$$

is space-like $(p)^2 = p^a p_a < 0$. A jet is a set of a few consecutive particles -usually less than 10- whose momenta are almost parallel and whose energies are much bigger than the gas average. Jets have a remarkable property: when particles in the same jet interact, their momentum is in general strongly boosted in the jet direction and their energies can increase even by $10 \div 30$ orders of magnitude in a single transition. Note that, as demonstrated in [12,13], Gott pairs cannot be created when $4GU < 1$.

Simple qualitative considerations show that a universe with spacelike total momentum is either exploding or imploding. Results obtained from our simulations of such universes always correspond to the imploding case, where particles looping around the whole universe always get boosted in the same direction as their velocity. In an exploding universe, particles with speed lower than the global expansion rate cannot complete a loop around the universe but, by construction, our algorithm does not take into account this global effect. The instability that we find for spatial total momentum is in agreement with the big-crunch phenomena studied in [6].

5.3. High energies. For a spacetime of topological type $S^2 \times \mathbb{R}$, the Gauss-Bonnet equality implies $4GU = 2$. Suppose that one or more particles are very far from all the others and do not interact with them; these particles can be neglected so that it makes sense to consider a particle system with energy $1 \leq 4GU < 2$. The particular case $4GU = 1$ also corresponds to a (globally static) cylindrical universe.

In the whole range $1 \leq 4GU \leq 2$ we observed the same behaviour as for spacelike total momentum: Gott pairs, jet formation and instability due to divergent energies. This suggests that, in our approximation, a closed universe usually develops Gott pairs.

Because of the finite precision of numerical simulations, it is impossible to constraint U to be exactly equal to a predefined value X. Using double precision floating point numbers (8 bytes) we can only obtain simulations with $|U - X| \lesssim 10^{-15}$ and U fluctuates in that range as the simulation proceeds. We have even seen, though very rarely, simulations with $4GU < 1$ explode due to $4GU$ becoming > 1 by finite precision errors. For this reason, it is not possible to distinguish the gas behaviour at $4GU = 1$ (or $4GU = 2$) from the behaviour at very near values of U. What we have actually seen is a strong instability in both cases.

6. Classical statistics. In the next section we shall compute the entropy of the gravitating gas when the energy satisfies $4GU < 1$ and we shall compare the outcome of the simulations with the entropy of a non-gravitating relativistic Boltzmann gas in (2+1) dimensions. Here we recall a few classical statistical mechanics results.

For a classical relativistic gas with 2-dimensional volume V, the Maxwell-Boltzmann's distribution f_0 takes the form

$$f_0(\vec{p}) \;=\; \left(\frac{N}{V}\right) \frac{e^{(\,m/kT)}}{2\,\pi\,k\,T\,(\,m+kT\,)} \;\exp\left(-\frac{\sqrt{(\vec{p})^2\,+\,m^2}}{kT}\right)$$

and has normalization

$$\int d^2x\, d^2p\; f_0(\vec{p}) \;=\; N \quad .$$

The pressure P can be computed in the usual microscopic way: one has to sum the variation of momenta over the particles hitting a unitary surface. The result is the same as for a perfect gas

$$PV \;=\; NkT \quad .$$

The internal energy turns out to be

$$U \;=\; U(T,V) \;=\; N\,\langle\, p^0\,\rangle \;=\; NkT\left(\frac{m+2kT}{m+kT}\right) \,+\, Nm \quad .$$

According to Boltzmann's H-theorem, the entropy is

$$S \;\equiv\; -k\int d^2x\, d^2p\, f_0 \log(f_0) \;=\; kN\left(\frac{m+2kT}{m+kT} - \log\frac{N}{V} + \log\Big(kT(m+kT)\Big)\right)$$

which satisfies the fundamental relation

$$\frac{\partial S(U,V)}{\partial U} \;=\; \frac{1}{T} \quad . \tag{6.1}$$

When $m = 0$ one obtains

$$U \;=\; U(T,V) \;=\; 2NkT \tag{6.2}$$

$$S \;=\; S(U,V) \;=\; kN\left(2 - \log\frac{N}{V} + 2\log\frac{U}{2N}\right) \quad . \tag{6.3}$$

Note that equation (6.2) is in agreement with the Equipartition Theorem because the dispersion relation takes the form $\sqrt{|\vec{p}\,|^2}$.

7. Statistical properties of (2+1) dimensional gravitating gas. Numerical simulations showed that the open universe with energy $4GU < 1$ is stable, i.e. the system always reaches a stationary state, therefore we shall now examine the statistical properties of such a gas. In particular, we will study the entropy and the temperature of the system.

In order to verify that the stationary state corresponds to thermal equilibrium, we have checked that the connected part of the 2-particle probability distribution

$$f_{2c}(p_i, p_j) = f_2(p_i, p_j) - f(p_i) \cdot f(p_j)$$

is numerically negligible. We again apply Boltzmann's H-theorem to give a numerically usable definition of entropy; considering only particle momenta, one has

$$S \equiv -k \int d^2p \; f(p) \; \log(f(p)) \quad . \tag{7.2}$$

For the plots, a "reduced" definition of S has been used:

$$\overline{S} = \frac{S}{Nk} - \log\left(\frac{2\pi}{N}\right) = -\int_m^{+\infty} dp^0 \rho(p^0) \log\left(\frac{\rho(p^0)}{p^0}\right) \quad .$$

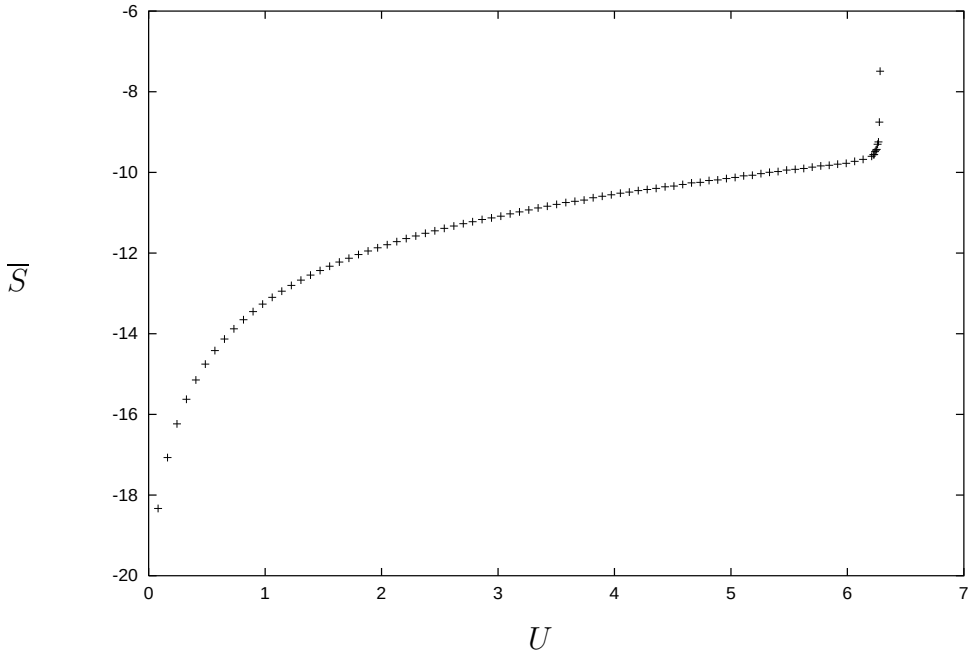

Figure 4: plot of $\overline{S}$ at $N = 1000,\ m = 0$

As one can deduce from Figure 4, the values of S that we found are very well fitted by $S \propto \log(U) + \text{const}$ for energies up to $4GU \lesssim 0.8$. As U grows, $\partial S/\partial U$ decreases more slowly than expected. At a certain value $U = \overline{U}$, $\partial S/\partial U$ stops decreasing and S has a flex $(\partial^2 S/\partial U^2 = 0)$. For $U > \overline{U}$, the second derivative of S is > 0 and increases as $4GU \to 1$. In the same limit, the entropy S seems to diverge. This behaviour is supported by the following argument. For a given U, let f_m be the

maximum value of $f(|\vec{p}|)$. At $4GU > 1$ we already said that simulations diverge, and then it is reasonable to assume that

$$\lim_{4GU \to 1} f_m = 0 \quad . \tag{7.3}$$

From the definition (7.2) of S we get

$$S \;=\; -k \int d^2p\; f(p)\; \log(f(p)) \;>\; -k\; \log(f_m) \int d^2p\; f(p) \;=\; -Nk\;\; \log(f_m)\;,$$

then by means of (7.3) one finds

$$\lim_{4GU \to 1} S(U) \;=\; +\infty \quad .$$

Finally, for $4GU > 1$ there is no equilibrium state, so S is not defined in that range.

7.1 Temperature The standard interpretation of the entropy behaviour as a function of U uses the relations

$$\frac{1}{T} \;=\; \frac{\partial S(U,V)}{\partial U} \tag{7.4}$$

and

$$\frac{1}{C} \;=\; -T^2 \frac{\partial^2 S(U,V)}{\partial U^2} \quad . \tag{7.5}$$

According to these equations, the temperature T has a maximum at $U = \overline{U}$ and the heat capacity C diverges at $U = \overline{U}$ and is negative at $U > \overline{U}$. Such results look rather strange as they violate the thermodynamical inequalities. In a standard statistical system with additive S and C, the violation of these inequalities implies instability. This is not true in our system; when $4GU < 1$ the gravitating gas reaches the equilibrium. The standard instability theorems do not apply to our system because S and C are not additive.

In order to illustrate the peculiar properties of the gravitating gas, consider for example two identical systems, each one in equilibrium, which are isolated. When these two systems are put in thermal contact, in general the resulting system is not in equilibrium.

The temperature T defined in equation (7.4) is not a convenient variable to deduce the stability properties of the gravitating system. For the Boltzmann gas, one can give different definitions of temperature that are all equivalent. In the case of a gravitating gas, these definitions are no longer equivalent. Firstly, we have two energies that characterize the equilibrium state: $N\mathcal{E} \;=\; N\langle\, p^0 \,\rangle$ and U. Due to the peculiar energy conservation law in (2+1) gravity, $N\mathcal{E}$ and U are not trivially related. Secondly, we shall introduce the temperature in two ways: by means of appropriate Equipartition Theorem (6.2) or by means of the entropy (6.1). For simplicity we shall concentrate

on the $m = 0$ case. By taking into account all the possibilities, the various definitions of temperature are:

$$T_1 \;=\; \frac{\mathcal{E}}{2k} \qquad , \qquad T_2 \;=\; \frac{U}{2Nk}$$

$$T_3 \;=\; N\left(\frac{\partial S}{\partial \mathcal{E}}\right)^{-1} \qquad , \qquad T_4 \;=\; \left(\frac{\partial S}{\partial U}\right)^{-1} \quad .$$

T_4 coincides with definition (7.4).

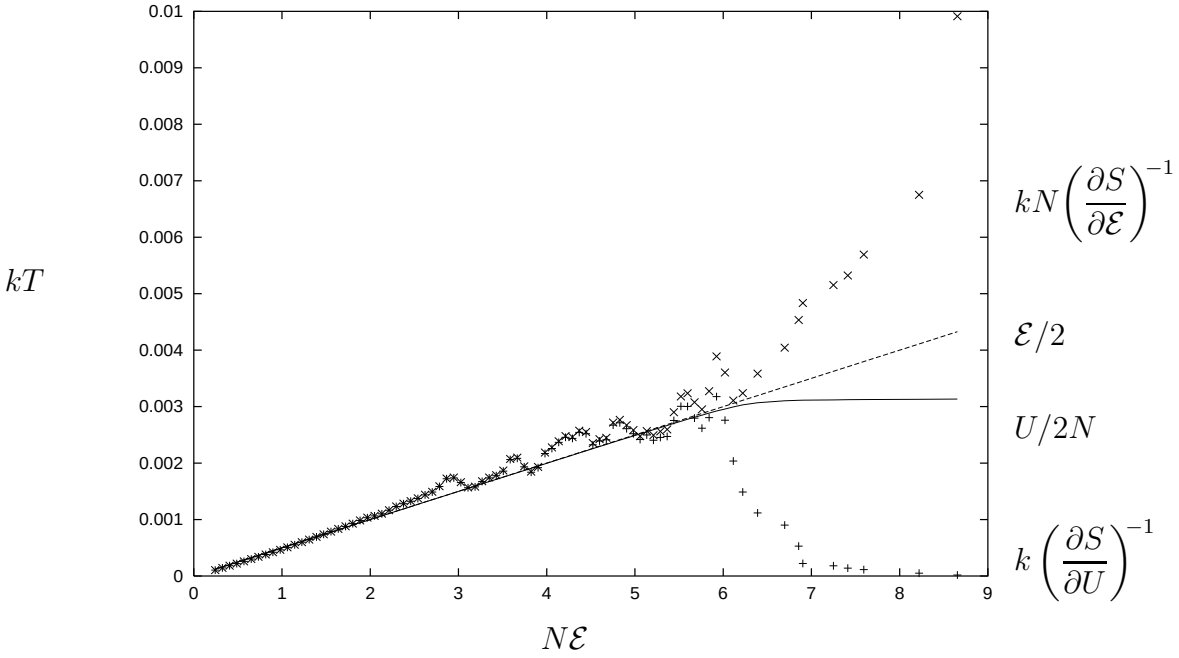

Figure 5: plot of kT at $N = 1000,\ m = 0$

As we can see from the plot, all the definitions of T are in good agreement with each other as long as $4GU \lesssim 0.8$, while at higher energies each definition of T has a different behaviour: when plotted as a function of $\mathcal{E}$, $T_3 = N(\partial S/\partial \mathcal{E})^{-1}$ grows more than linearly, while $T_4 = (\partial S/\partial U)^{-1}$ has the maximum we already anticipated, then drops quite quickly. The other two ones $T_1 = \mathcal{E}/2k$ and $T_2 = U/2Nk$ have quite obvious behaviours: the first one is linear, while the second asymptotically approaches $1/8GNk$ as $4GU \to 1$.

By plotting S as function of $\log \mathcal{E}$ (see Figure 6), one notices that the fit $S \propto \log \mathcal{E}$ is quite reasonable at all energies. This suggests that T_3 is in some sense the most suitable definition of temperature. With respect to $T = T_3$, the $4GU \to 1$ limit corresponds to $\mathcal{E} \to +\infty$ and simultaneously $T \to +\infty$; moreover, the heat capacity is always finite and positive.

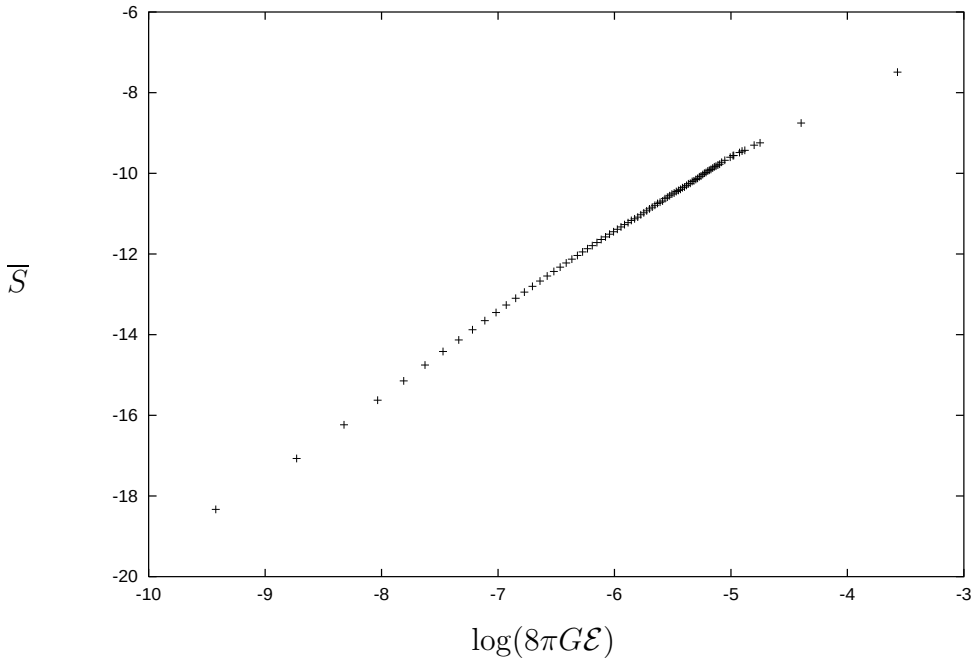

Figure 6: plot of $\overline{S}$ at $N = 1000,\ m = 0$

8. Conclusions. A system of gravitating particles in (2+1) dimensions has peculiar thermodynamic properties due to the non-Abelian structure of the energy-momentum conservation law. Depending on the total 3-momentum, numerical simulations show different system behaviours. When the internal energy U satisfies $4GU < 1$, an equilibrium state is reached; on the other hand, for space-like total momentum or when $4GU > 1$, dynamical instabilities appear. This is consistent with the interpretation of particle diverging energies as a big-crunch phenomenon.

In the presence of thermal equilibrium, we started with the distribution function in momentum space and studied the thermodynamic properties of the gravitating particles. In the low energy limit, which is equivalent to the low gravity limit, the system behaves as a Boltzmann relativistic gas. For $4GU \to 1$, the entropy and the mean value of the kinetic energy seem to diverge.

Starting from the various equivalent definitions of temperature in Boltzmann gas, we have considered their behaviour also in the energy range of large gravity effects. These temperatures turn out to be different. One of them is well behaved also in the $4GU \to 1$ limit and does not violate the thermodynamic inequalities; in particular, the associated heat capacity is always finite and positive.

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